# Part II: Feature Extraction

A significant use case for MRI is the identification of neurodegenerative diseases and tumours within the brain. MRI scans produce enormous amounts of data which can be extremely difficult to analyse manually. As such, a combination of mathematical and machine learning techniques are often employed to aid such analysis. We will demonstrate one of the common mathematical techniques used within this field. For practicality’s sake, black and white images of human faces will be used as a demonstrative proxy for brain cross-section images. Where while in the latter context, we would be interested in identifying tumours and/or biomarkers of disease, in the context of our proxy we will simply be attempting to detect moustaches. In a real implementation, machine learning would most likely be used in conjunction with the mathematical techniques explained below, however since we will be mainly focusing on the mathematics, our detector will be highly rudimentary at best and will not rely on any machine learning techniques. The mathematical technique is explained as follows:

Say we have a set of images of human faces denoted as matrices . Our goal is to use this data to derive some method for identifying moustaches within the images. The matrices are first vectorized, i.e., their columns are stacked on top of one another to form vectors:

where . The resultant vectors are then stacked column-wise to form a matrix containing the data from all images:

We will identify the “average face” by taking the column-wise mean of :

This can be used to mean centre :

where is a column vector of ones. We will now consider the reduced singular value decomposition of :

where , and . This can be visualised as follows:

where , and and denote the columns of and respectively. The RHS can be rearranged to form a sum of rank one matrices:

Since by construction we have that , we also have that . As such, we can see that the relative contribution of each matrix towards the reconstruction of is determined solely by the value of . Since , it can be deduced that said contribution monotonically decreases as increases. It turns out that in most instances this happens very quickly. As such, can often be very well approximated by

where or even . Returning to matrix format this can be expressed as

where , and . From this we can write an equation for the column of :

where is the row of or more intuitively the column of .

The columns of are called eigenfaces. They represent the principal components of the data. More specifically they are the vectors that minimise total squared reconstruction error,

or equivalently,

for all . If the system is underdetermined, that is, , is simply the minimum-norm solution out of all valid orthonormal bases.

Recall that:

Let . Thus, the expression above can be rewritten as:

One can clearly see that is simply the coordinate vector of with respect to the basis . Note that is simply a projection of onto the lower rank bearing basis . Now let’s say we have a new vectorized image and we want to find its closest representation with respect to the basis . To do this one would simply find the projection of onto with respect to . Since is orthonormal, said projection is given by:

This can then be expressed in terms of the standard basis as follows: